Internal dynamics of a vector soliton in a nonlinear optical fiber

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We analyze the dynamics of a vector soliton governed by a nearly integrable system of coupled nonlinear Schrödinger equations. Inserting a Gaussian ansatz into the Lagrangian density, we derive a system of ordinary differential equations for the evolution of the ansatz parameters. We find a continuous family of stationary solutions to these equations which can be interpreted as vector solitons with an arbitrary polarization. Examining small internal vibrations of the vector soliton, we find three eigenmodes, of which only two were previously known. The additional internal oscillation eigenmode gives rise to antisymmetric oscillations of the symmetric soliton (45° polarization). We also find the small-vibration eigenmodes for arbitrary polarization, though in an implicit form. Additionally, we find a threshold value of the relative velocity of the two polarizations that leads to splitting of the vector soliton for arbitrary polarization.

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I. INTRODUCTION

The fast-approaching major engineering application that features solitons as the medium of telecommunication transmission continues to evoke intense interest in the characterization of soliton's behavior in a variety of real-world situations. For their use as information bits in a fiber-optic communication system one would ideally wish the governing equations to be completely integrable, making the bits true solitons, hence not subject to perturbation, decay, or other forms of signal degradation. The tensor character of the $\chi^{(3)}$ nonlinear susceptibility, however, generally leads to unequal coefficients of nonlinear self-phase and cross-phase modulation ($\beta\neq 1$ in our notation below), destroying integrability. Whereas the completely integrable system is well understood, the nonintegrable system has a qualitatively more complex behavior and is more difficult to analyze.

Vector solitons in nonintegrable models may exhibit, for example, the following effects not found in completely integrable systems. Two colliding vector solitons will give rise to radiation [1,2]. Below some threshold of relative velocity and ratio of amplitudes, colliding solitons may interact so as to either merge into a single soliton or destroy each other in the collision process [1,3]. On an even more basic level, vector solitons do not possess the characteristic (for true solitons) hyperbolic secant shape; rather, they are subject to nonsymmetries between the two polarization modes [1]. Each polarization mode takes on a slightly asymmetric form, with the tails in particular exhibiting complex behavior [1]. Additionally, vector solitons near equilibrium are subject to a frequency chirp and oscillations about the minimum potential point [4,5].

Though computational studies have explored the mathematical terrain (see, for example, Ref. [1]), useful analytic results have been found only in a few relatively simple instances. The completely integrable limit ($\beta=1$) may be solved by the inverse scattering transform [6]. When the system is not integrable, however, even stationary vector solitons are known in a closed form only when the amplitude of one of the polarizations is zero or when the two amplitudes are equal (though there are exact solutions for arbitrary polarization in other closely related systems [7]). Numerical simulations have yield many interesting dynamical results, though of course not with the universality provided by analytic methods. Some approximate analytic results have been found for the vector soliton dynamics when the initial conditions are symmetric with respect to the two polarization modes [5].

In the current work, by first taking one step backward in employing an oversimple approximation of the vector solitons' form, we have been able to take two steps forward in providing alternative qualitatively accurate analytic descriptions of the vector solitons' dynamics. We make the assumption that vector solitons maintain a prescribed wave form, in this case one with a Gaussian shape. We choose this form because it is the only one that allows us to solve all the necessary integrals in the general case, in which the vector soliton is not presumed symmetric. Furthermore, the Gaussian shape is not really that different from a sech function, differing only in the shape of the tail. (If one matches amplitudes and curvatures at the top.) Earlier work has been done using a Gaussian shape [8] and the qualitative results were essentially identical with those [4] for a sech shape. Even the qualitative results were essentially equivalent, with each model providing a good description of the evolution of the pulse. We allow all of our pulse parameters—position, phase, width, amplitude, frequency, and chirp—to vary independently in each polarization. Substituting such an assumed form of the solution into the Lagrangian density reduces the number of variables from an infinite number to an finite one. Integration of the Lagrangian density over time reduces the problem from a partial differential equation in two dimensions to an ordinary differential equation (ODE) in the spatial dimension alone. The system is sufficiently tractable at this point that we may derive analytically a number of interesting dynamical properties of the system.

Ueda and Kath [5] have used the same technique, but with hyperbolic-secant-shaped pulses, as opposed to our Gaussian-shaped pulses. Though the former is in principal more accurate, Ueda and Kath were forced to artificially set the pulse widths in the two polarizations to be equal in order to make the equations tractable. Our model, though it may give greater quantitative error, allows for a more complete qualitative description of the vector solitons' behavior. Also important is that this model may be extended much more easily to the description of interacting vector solitons. (Analysis of collisions between vector solitons, however, is deferred to another work.)

II. FORMALISM

Vector solitons in an optical fiber are governed by a set of nondimensionalized coupled nonlinear Schrödinger equations,

$$iu_x+i\delta u_t+\tfrac{1}{2}u_{tt}+(|u|^2+\beta|v|^2)u=0\ , \tag{1a}$$

$$iv_x-i\delta v_t+\tfrac{1}{2}v_{tt}+(\beta|u|^2+|v|^2)v=0\ , \tag{1b}$$

where the pulse width is the picosecond range, group-velocity dispersion is negative (anomalous), the birefringence is not too small, and a host of more obvious assumptions such as negligible absorption, homogeneity, etc., are implied. u and v are the nondimensionalized amplitudes of the two polarization modes. The constant β is a fiber parameter which is in the range $\{\frac{2}{3}\le\beta\le 2\}$, the lower limit representing linearly birefringent fibers, the upper limit circularly birefringent ones. The linear-birefringence term δ may be eliminated by the transformation

$$u\to ue^{i[-\delta t+(1/2)\delta^2x]}\ , \tag{2a}$$

$$v\to ve^{i[\delta t+(1/2)\delta^2x]} \tag{2b}$$

and can therefore be set to zero without loss of generality, which we do henceforth.

Inspection of exactly known limiting cases guides the selection of a suitable ansatz. The simplest limiting case is that in which there is no interaction between the modes, either because the coupling term in the governing equations β, is nil or because all of the energy in a given region is in one of the modes:

$$u=\eta\,\mathrm{sech}[\eta(t-Vx)]e^{i[Vt+(1/2)(\eta^2-V^2)x]}$$
$$\equiv u_{\mathrm{NLS}}(x,t)\ , \tag{3a}$$

$$v=0\ . \tag{3b}$$

Manakov's equations [6] ($\beta=1$) have one-soliton solutions in the form of the nonlinear-Schrödinger-equation (NSL) soliton but with arbitrary polarization:

$$u=e^{i\phi_u}\cos(\alpha)u_{\mathrm{NLS}}(x,t)\ , \tag{4a}$$

$$v=e^{i\phi_v}\sin(\alpha)u_{\mathrm{NLS}}(x,t)\ . \tag{4b}$$

For general β there is a vector soliton polarized at 45°:

$$u=v=\eta\,\mathrm{sech}[\sqrt{1+\beta}\,\eta(t-Vx)]$$
$$\times\exp i\{Vt+\tfrac{1}{2}[(1+\beta)\eta^2-V^2]x\}\ . \tag{5}$$

The ansatz that we choose in this work is

$$u=\eta_u\exp\left[-\frac{1}{2}\left[\frac{t-y_u}{W_u}\right]^2\right]$$
$$\times\exp i\left[\sigma_u+2V_u(t-y_u)+\frac{b_u}{2W_u}(t-y_u)^2\right]\ , \tag{6a}$$

$$v=\eta_v\exp\left[-\frac{1}{2}\left[\frac{t-y_v}{W_v}\right]^2\right]$$
$$\times\exp i\left[\sigma_v+2V_v(t-y_v)+\frac{b_v}{2W_v}(t-y_v)^2\right]\ . \tag{6b}$$

This assumed form of the solution allows us to independently vary, for each polarization mode, the central position, pulse width, and amplitude; there is also an independent constant phase, carrier frequency, and frequency chirp for each mode. Letting these parameters vary dynamically under the obeisance of Eqs. (1) includes, among some of the most important effects, internal energy excitations, pulsation of the vector solitons' shape, and attraction between the two modes. Not included is radiation, asymmetries in the individual modes, and anything but very simple changes in the pulse shape. We have used Gaussian pulses rather than hyperbolic secant ones, which would agree better with the limiting cases and are presumably more accurate. The reason is simply that Gaussian pulses, being easier to handle mathematically, give tractable results for a variety of situations in which hyperbolic secants do not. The bottom line is that using Gaussian pulses sacrifices some quantitative accuracy in exchange for a more complete qualitative picture.

III. EVOLUTION EQUATIONS AND CONSERVED QUANTITIES

Substituting Eqs. (6) into the Lagrangian density from which the governing equations may be derived,

$$\mathcal{L}=\frac{i}{2}(u_xu^*-uu_x^*)+\frac{i}{2}(v_xv^*-vv_x^*)-\tfrac{1}{2}(|u_t|^2+|v_t|^2)$$
$$+\tfrac{1}{2}|u|^4+\tfrac{1}{2}|v|^4+\beta|u|^2|v|^2\ , \tag{7}$$

integrating the result over time (t) from $-\infty$ to ∞, and taking the variation with respect to each of the parameters in the ansatz gives a simplified system of equations of motion.

We get, as a result, a six-dimensional phase space and four conserved quantities. We express the system as a set of six first-order equations (it may also be written as three second-order equations):

$$\frac{d}{dx}\Delta y = \Delta V \ , \tag{8a}$$

$$\frac{d}{dx}\Delta V = -\frac{8}{\sqrt{\pi}}\beta(M_u + M_v)\frac{\Delta y}{(W_u^2+W_v^2)^{3/2}} \times \exp\left[\frac{-\Delta y^2}{W_u^2+W_v^2}\right] , \tag{8b}$$

$$\frac{d}{dx}W_u = 2b_u \ , \tag{8c}$$

$$\frac{d}{dx}(2b_u) = -\frac{4}{W_u}\left[\frac{M_u}{\sqrt{2\pi}W_u} - \frac{1}{W_u^2} + BM_v W_u^2\right] , \tag{8d}$$

$$\frac{d}{dx}W_v = 2b_v \ , \tag{8e}$$

$$\frac{d}{dx}(2b_v) = -\frac{4}{W_v}\left[\frac{M_v}{\sqrt{2\pi}W_v} - \frac{1}{W_v^2} + BM_u W_v^2\right] , \tag{8f}$$

where

$$B \equiv 2\beta[\pi(W_u^2+W_v^2)^3]^{-1/2}\left[1 - 2\frac{\Delta y^2}{W_u^2+W_v^2}\right] \times \exp\left[\frac{-\Delta y^2}{W_u^2+W_v^2}\right] ,$$

$$\Delta y \equiv y_u - y_v \ ,$$

and

$$\Delta V \equiv V_u - V_v \ .$$

The four constants of integration are the energy in each mode,

$$M_u = \sqrt{\pi}\eta_u^2 W_u \ , \tag{9a}$$

$$M_v = \sqrt{\pi}\eta_v^2 W_v \ , \tag{9b}$$

the total momentum,

$$P = M_u V_u + M_v V_v \ , \tag{10}$$

and the Hamiltonian,

$$H = \frac{8}{M_u}(\tfrac{1}{4}M_u b_u)^2 + \frac{M_u}{2}\left[\frac{M_u}{\sqrt{2\pi}} - \frac{1}{W_u}\right]^2 + \frac{8}{M_v}(\tfrac{1}{4}M_v b_v)^2 + \frac{M_v}{2}\left[\frac{M_v}{\sqrt{2\pi}} - \frac{1}{W_v}\right]^2 + \frac{4}{\mu}(\mu[V_u - V_u])^2 - \frac{2\beta}{\sqrt{\pi}}\frac{M_u M_v}{\sqrt{W_u^2+W_v^2}}\exp\left[\frac{-\Delta y^2}{W_u^2+W_v^2}\right] , \tag{11}$$

where μ is the reduced mass,

$$\frac{1}{\mu} \equiv \frac{1}{M_u} + \frac{1}{M_v} \ .$$

To within a constant, the frequency variables V are the conjugate momenta of the positions y; the frequency chirps b are the conjugate momenta of the pulse widths W. Three of the conserved quantities, the two energies and the Hamiltonian, play important parts in the dynamics, but the fourth, momentum, drops out. Obviously event this new system of ordinary differential equations, Eqs. (8), is nontrivial to solve in general. But, as we show presently, it allows us to surmount several difficulties that had made the case of $M_u \neq M_v$ intractable in earlier studies.

IV. INTERNAL MODES OF THE VECTOR SOLITON

Equations (8) possess, for any values M_u and M_v, exactly one fixed point:

$$\Delta y = \Delta V = b_u = b_v = 0 \ , \tag{12a}$$

$$W_u = \frac{\sqrt{2\pi}}{M_u}\left[1 + \left[\frac{2}{1+\alpha^2}\right]^{3/2}\beta\frac{M_v}{M_u}\right]^{-1} , \tag{12b}$$

$$W_v = \frac{\sqrt{2\pi}}{M_v}\left[1 + \left[\frac{2}{1+\alpha^{-2}}\right]^{3/2}\beta\frac{M_u}{M_v}\right]^{-1} = \alpha W_u \ , \tag{12c}$$

where α is defined by the equation

$$\beta\left[\frac{2}{1+\alpha^2}\right]^{3/2}\left[\alpha^4 - \frac{M_v}{M_u}\right] + \frac{M_v}{M_u}\alpha - 1 = 0 \ . \tag{12d}$$

[Note that Eq. (12d) has exactly one solution on the positive real axis.] This solution may be regarded as representing a unique stationary vector soliton with arbitrary polarization [the polarization angle is equal to $\tan^{-1}(M_v/M_u)$]. We will demonstrate below that these vector solitons are stable within the framework of the approximation employed.

In the general case, the widths of the two components of the vector soliton are given by Eqs. (12) in an implicit form. They may be expressed explicitly either when the energy is entirely in one polarization mode (which is trivial), or when the energy is split evenly between the modes. Here, we will analyze only the case when the polarization angle is near 45°. The case of a general polarization is essentially the same except it is much more technically complex. We expand Eqs. (12) about this polarization angle, and Eqs. (12b) and (12c) can then be brought into the following form:

$$W_u = \frac{\sqrt{2\pi}}{(1+\beta)M}\left[1 + \frac{(\beta-1)}{1+4\beta}\frac{\delta M}{M}\right] , \tag{13a}$$

$$W_v = \frac{\sqrt{2\pi}}{(1+\beta)M}\left[1 - \frac{(\beta-1)}{1+4\beta}\frac{\delta M}{M}\right] , \tag{13b}$$

where $M_u \equiv M + \delta M$ and $M_v \equiv M - \delta M$, $\delta M / M \ll 1$. We now expand and linearize Eqs. (8) about the fixed point represented by Eq. (13). One obtains three second-order ODE's for the variations of Δy, W_u, and W_v about the fixed point. The equation for $\delta\Delta y$ (we use δ to indicate a variation) separates from those for δW_u and δW_v and contains only one mode of oscillation. The equations for δW_u and δW_v couple only these degrees of freedom and contain two modes of oscillations. In general, the modes are a complex mixture of symmetric oscillations, in which the polarization modes' widths dilate and contract together, and antisymmetric oscillations, in which one polarization mode expands while the other narrows. Two corresponding eigenmodes and their frequencies are given by the eigenvectors and the eigenvalues of the matrix, which are obtained by linearization of Eqs. (8c)–(8f):

$$\begin{bmatrix} -\dfrac{1}{W_u^4}-\dfrac{6}{\sqrt{\pi}}\dfrac{\beta M_v W_v^2}{(W_u^2+W_v^2)^{5/2}} & \dfrac{6}{\sqrt{\pi}}\beta\dfrac{M_v M_u W_v}{(W_u^2+W_v^2)^{5/2}} \\ \dfrac{6}{\sqrt{\pi}}\beta\dfrac{M_u W_u W_v}{(W_u^2+W_v^2)^{5/2}} & -\dfrac{1}{W_v^4}-\dfrac{6}{\sqrt{\pi}}\beta\dfrac{M_u W_u^2}{(W_u^2+W_v^2)^{5/2}} \end{bmatrix}, \tag{14}$$

where the basis is the vector

$$\begin{bmatrix} \delta W_u \\ \delta W_v \end{bmatrix}.$$

The expression (14) is valid for any value of M_u/M_v. Close to $M_u=M_v$, the frequencies and their eigenvectors may be found explicitly:

$$\omega_+=\frac{1}{\pi}(1+\beta)^2M^2+O(\delta M^2)\ , \tag{15a}$$

$$\begin{bmatrix} 1-\dfrac{1}{2}\left[1-\dfrac{(1+2\beta)(13\beta-8)}{3\beta(1+4\beta)}\right]\dfrac{\delta M}{M} \\ 1+\dfrac{1}{2}\left[1-\dfrac{(1+2\beta)(13\beta-8)}{3\beta(1+4\beta)}\right]\dfrac{\delta M}{M} \end{bmatrix}, \tag{15b}$$

and

$$\omega_-=\frac{1}{\pi}(1+\beta)^{3/2}(1+4\beta)M^2+O(\delta M^2)\ , \tag{16a}$$

$$\begin{bmatrix} 1-\dfrac{1}{2}\left[1+\dfrac{(1+2\beta)(13\beta-8)}{3\beta(1+4\beta)}\right]\dfrac{\delta M}{M} \\ -1-\dfrac{1}{2}\left[1+\dfrac{(1+2\beta)(13\beta-8)}{3\beta(1+4\beta)}\right]\dfrac{\delta M}{M} \end{bmatrix}. \tag{16b}$$

The third oscillation mode is that of the relative position of the two components of the vector soliton Δy, which has the frequency

$$\omega_{\Delta y}=\frac{1}{\pi}\beta^{1/2}(1+\beta)^{3/2}M^2+O(\delta M^2)\ . \tag{17}$$

All three vibrational frequencies are real to first order in $\delta M/M$, i.e., the vector soliton is stable at least at this order. It is also noteworthy that, to first order in $\delta M/M$, the two dilation-contraction modes couple together, but neither couples to the relative position oscillation. Provided that the interaction between the polarizations is nonzero ($\beta>0$), the first ("+") mode, Eqs. (15), will be purely symmetric at $M_u=M_v$ but will acquire an antisymmetric part for $M_u\neq M_v$; the second ("−") mode, Eqs. (16), will be purely antisymmetric at $M_u=M_v$ but will acquire a symmetric part for $M_u\neq M_v$. The smaller is the coupling between modes, β, the stronger is the distortion of the eigenvector produced by a small energy asymmetry δM.

The calculation of the second ("−") vibrational mode, Eqs. (16), which rounds out the vibrational analysis, has not to our knowledge been previously published. The other oscillation modes have been calculated previously, but only at $M_u=M_v$, not including the first-order perturbations nor the more general nonsymmetric case. Finally, note that, *mutatis mutandis*, each of the symmetric parts of our results agree up to a constant with the results of Ueda and Kath [5], who calculated two of the vibrational frequencies for symmetric hyperbolic-secant pulses. (Ueda and Kath's $\omega_{\Delta y}$ is $\pi/\sqrt{15}$ times ours and their ω_+ is identical to ours.)

As a closing remark to this section, we point out that the stability found here will only be valid as long as the radiation modes remain frozen out. Considering the particular case with the zero-polarization angle, Eqs. (8c) and (8d) reduce to

$$\frac{d^2}{dx^2}W_u=-\frac{4}{W_u^3}\left[\frac{M_uW_u}{\sqrt{2\pi}}-1\right],$$

which has the fixed point

$$M_uW_u=\sqrt{2\pi}\ . \tag{18}$$

Actually, Eq. (18) represents the well-known amplitude-width relation for the soliton governed by *one* nonlinear Schrödinger equation. If the amplitude and width of the soliton do not satisfy Eq. (18), the evolution governed by the full partial differential equation [Eq. (1a) with $\beta=0$] would show the shape (amplitude-width) oscillations of the soliton [4], which, however, would gradually fade because of emission of radiation. The soliton will thus slowly readjust its shape, radiating off excess energy, and will asymptotically reach a stationary shape satisfying Eq. (18). Quite similarly, the shape oscillations of the vector soliton governed by the coupled equations (1) are subject to radiative damping. However, one could not expect to see this effect in the framework of the approximation employed above, simply because the radiative modes had been completely frozen out from Eq. (6).

V. DYNAMICS OF ESCAPING SOLITONS

Also important to know is the separatrix between bound solitons and splitting ones. The interaction term in the governing equations creates an effective attraction between the pulses in the two polarization modes. Thus there is a minimum kinetic energy required for the pulse to split apart.

The minimum possible value of the Hamiltonian, Eq. (11), for widely separated pulses is zero. Therefore zero is the minimum value of the Hamiltonian necessary for complete pulse separation. (This, however, does not guarantee splitting of the vector soliton in all cases, since the pulses may contain internal energy, chirps and/or nonequilibrium widths, which may subtract enough of the kinetic energy to reduce it to below the escape threshold.) The minimum relative velocity required at the point of zero separation to reach the escape threshold, i.e., the escape velocity, can be found by equating the Hamiltonian to zero at the point of zero pulse separation where all the terms except the kinetic energy are minimized (which implies, in particular, $b_u = b_v = 0$):

$$\Delta V^2_{\text{escape}} = \left[\frac{1}{M_u} + \frac{1}{M_v} \right] \left[\frac{\beta M_u M_v}{2\sqrt{\pi(W_u^2 + W_v^2)}} - \frac{M_u}{8} \left[\frac{M_u}{\sqrt{2\pi}} - \frac{1}{W_u} \right]^2 - \frac{M_v}{8} \left[\frac{M_v}{\sqrt{2\pi}} - \frac{1}{W_v} \right]^2 \right] , \tag{19a}$$

where W_u and W_v are taken from Eqs. (12). Near $M_u = M_v$ this gives

$$\Delta V^2_{\text{escape}} = \frac{1}{2\pi} \beta (1 + \tfrac{1}{2}\beta) M^2 + O(\delta M^2) . \tag{19b}$$

If there is another contribution to the Hamiltonian of the vector solitons, such as chirp, nonequilibrium width, or pulse separation, the velocity needed to reach the separation threshold will be lower. Thus the escape velocity is neither the minimum velocity which ensure escape nor the maximum below which the vector soliton remains in one piece, but may rather be viewed as a guidepost which marks a grey area between the two regions.

A vector soliton that has held together over one or more oscillations may still split apart in the future if and when the system wanders into a region of phase space where Δy is large and most of the Hamiltonian comes from its kinetic part rather than its internal part. This is cause for caution in the interpretation of numerical simulations of pulse splitting, since there is no apparent way to tell *a priori* how many oscillations the system is going to perform before the vector soliton splits.

Ueda and Kath [5] computed the escape velocity similarly, but for hyperbolic-secant pulses at a 45° polarization. Their escape velocity, after making all appropriate adjustments, is equal to that in Eq. (19b) times $\sqrt{\pi/3}$. A similar analysis of the splitting problem was done by Kivshar [9], who used the sech ansatz for the vector soliton. For Manakov's system, some analysis based on the inverse scattering transform has been recently done in Ref. [10].

VI. CONCLUSION

By assuming that pulses in each polarization mode have a Gaussian shape, we were able to make advances in the qualitative analysis of the internal dynamics of a vector soliton in nonintegrable models of the birefringent optical fibers. Our analytical approach allows a description of the dynamics in the nonsymmetric case. Our results imply that there must exist a stable vector soliton with a stationary shape for *any* polarization, not only for the obvious cases of the 0°, 45°, and 90° polarizations. Within the framework of our approximation, we have found the eigenfrequencies of all the three internal modes of the vector soliton, of which, thus far, only two had been known (and only for the 45° polarization). The additional mode is the one that is responsible for the antisymmetric shape oscillations of the symmetric vector soliton. The threshold value of the relative velocity of the two polarizations that leads to the splitting of the vector soliton into unipolarized pulses has also been found for the general nonsymmetric case.

[1] C. R. Menyuk, J. Opt. Soc. Am. B **5**, 392 (1988).

[2] B. A. Malomed, J. Opt. Soc. Am. B **9**, 2075 (1992).

[3] B. A. Malomed and S. Wabnitz, Opt. Lett. **16**, 1388 (1991).

[4] D. Anderson, M. Lisak, and T. Reichel, J. Opt. Soc. Am. B **5**, 207 (1988).

[5] T. Ueda and W. K. Kath, Phys. Rev. A **42**, 563 (1990).

[6] S. V. Manakov, Zh. Eksp. Teor. Fiz. **65**, 505 (1973) [Sov. Phys. JETP **38**, 248 (1974)].

[7] D. N. Christodoulides and R. I. Joseph, Opt. Lett. **13**, 53 (1988).

[8] D. Anderson, Phys. Rev. A **27**, 3135 (1983).

[9] Yu. S. Kivshar, J. Opt. Soc. Am. B **7**, 2204 (1990).

[10] D. J. Kaup and B. A. Malomed, Phys. Rev. A **48**, 599 (1993).